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Superfield algorithm for higher order gauge field theories

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Abstract: We propose an algorithm for the construction of higher order gauge field theories from a superfield formulation within the Batalin-Vilkovisky formalism. This is a generalization of the superfield algorithm recently considered by Batalin and Marnelius. This generalization seems to allow for non-topological gauge field theories as well as alternative representations of topological ones. A five dimensional non-abelian Chern-Simons theory and a topological Yang-Mills theory are treated as examples.

Keywords: BRST Quantization, Topological Field Theories, Gauge Symmetry.

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1. Introduction

The superfield algorithm introduced by Batalin and Marnelius [1, 2] provides a general algorithm for constructing a class of first order gauge field theories. The Batalin-Vilkovisky (BV) formalism [3, 4] is there used as a framework for generating consistent quantum gauge field theories (at least up to gauge fixing and renormalization). In line with AKSZ [5], topological gauge field theories are naturally incorporated in the class of theories generated by this superfield algorithm. For instance, four and six dimensional Schwarz-type topological field theories which were constructed by means of the superfield algorithm in [6].

In this communication we propose a generalization of the superfield algorithm to allow for higher order gauge field theories. This is mainly achieved by introducing non-dynamical super multiplier fields. This generalized version may also allow for non-topological theories, suggested by our construction of a five dimensional Chern-Simons theory. All higher dimensional ($n \geq 5$) Chern-Simons theories generically possess local degrees of freedom according to [7, 8]. Possible theories are generated from a master action in superspace which is obtained by a ghost number prescription and a simple local master equation, which in turn is a generalization of the one considered in [1, 2]. The ordinary fields and antifields are components of the superfields, such that original gauge theories are obtained by a straightforward reduction procedure. To obtain a consistent quantum gauge field theory, a gauge fixing of the master action is necessary in addition to other issues such as for example renormalization. These issues are not treated in this paper.

In the first section 1 we introduce a generalized master action and the master equation that follows from it. We consider the boundary conditions that this master action has to satisfy in order to define a possible consistent gauge field theory. A key ingredient in the superfield algorithm considered in [1, 2] is the above mentioned local master equation

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obtained from the ordinary BV master equation in superfield form. Also here we consider a local master equation which is a generalization of the one in [1, 2] and we propose a generalized superfield algorithm for generating higher order gauge field theories as solutions to this new local master equation. Section 2 gives a description of the proposed generalized superfield algorithm and the motivation for its ingredients. In section 3 we construct a topological Yang-Mills theory by means of this generalized superfield algorithm. Section 4 and 5 continue by discussing the construction of three and five dimensional non-abelian Chern-Simons theories.

We use the same notation and conventions as the ones considered in [6].

2. General considerations

Consider a master action Σ written as a field theory living on a $2n$-dimensional supermanifold $\mathcal{M}$, with n Grassmann odd and n Grassmann even dimensions,

$$\Sigma = \int_{\mathcal{M}} d^n u d^n \tau \mathcal{L}_n(u,\tau)\,. \tag{2.1}$$

The supermanifold $\mathcal{M}$ is coordinatized by (u^a, τ^a), where $a = \{1, \ldots, n\}$ and u^a denotes the Grassmann even and τ^a the Grassmann odd coordinates respectively. The lagrangian density $\mathcal{L}_n$ is chosen to have the form

$$\mathcal{L}_n(u,\tau) = K^*_P(u,\tau) D K^P(u,\tau)(-1)^{\epsilon_P + n} - S(K^*_P, K^P, DK^*_P, DK^P)\,. \tag{2.2}$$

$K^P(u,\tau)$ represents the superfields and $K^*_P(u,\tau)$ the associated superfields, defined by (2.2) and the properties below. S is local in the superfields and D is the de Rham differential, $D^2 = 0$, with

$$D := \tau^a \frac{\partial}{\partial u^a}\,. \tag{2.3}$$

The chosen lagrangian is the same as the one considered in [1, 2], but here we allow terms in S involving the derivatives DK, DK^*. K^P and K^*_P denote the superfields and associated superfields but they also represent the super multiplier fields Λ^P, Λ^*_P, which are restricted by $D\Lambda^P = 0$ and $D\Lambda^*_P = 0$. Thus they are non-dynamical fields entering in the interaction term S, but not in the kinetic term $K^*_P D K^P$. A motivation for these properties of the multiplier fields is given in section 3.

The Grassmann parities of the associated superfields are

$$\epsilon(K^*_P) = \epsilon_P + n + 1 \qquad \text{where} \qquad \epsilon(K^P) := \epsilon_P\,. \tag{2.4}$$

This follows from the master action Σ, since this should be an even functional. The Grassmann parities of the super multiplier fields can be identified once the master action Σ has been specified. The conditions are that Σ have ghost number zero and the odd coordinate τ carry ghost number one and this implies ghost number restrictions on the superfields and corresponding associated superfields,

$$gh_{\#} K^P + gh_{\#} K^*_P = n - 1\,. \tag{2.5}$$

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It also follows that the local function S carries ghost number and Grassmann parity,

$$gh_{\#}S = n\,, \qquad \epsilon(S) = n\,. \tag{2.6}$$

The cornerstone of the BV-formalism is the classical master equation, requiring the master action Σ to satisfy,

$$(\Sigma, \Sigma) = 0\,, \tag{2.7}$$

where the bracket is the standard BV-bracket. Here this master equation may be written in a superfield form since the standard BV-bracket may be defined by

$$(F, G) := \int_{\mathcal{M}} F \frac{\overleftarrow{\delta}}{\delta K^P(u,\tau)} (-1)^{(\epsilon_P n)} d^n u\, d^n \tau \frac{\overrightarrow{\delta}}{\delta K^*_P(u,\tau)} G - (F \leftrightarrow G)(-1)^{(\epsilon(F)+1)(\epsilon(G)+1)}\,, \tag{2.8}$$

where F and G are two functionals of the superfields. Decomposing the superfields into ordinary fields by expanding in the odd coordinates τ^a this reduces to the standard expression for the BV-bracket [1].

It follows that this functional bracket has the standard properties, such as the graded symmetry property

$$(F, G) = -(-1)^{(\epsilon(F)+1)(\epsilon(G)+1)}(G, F)\,, \tag{2.9}$$

the graded Leibniz rule

$$\begin{aligned} (F, GH) &= (F, G)H + G(F, H)(-1)^{\epsilon(G)\epsilon(H)}\,, \\ (FG, H) &= (F, H)G + F(G, H)(-1)^{\epsilon(F)\epsilon(G)}\,, \end{aligned} \tag{2.10}$$

and the graded Jacobi-identity

$$((F, G), H)(-1)^{(\epsilon(F)+1)(\epsilon(H)+1)} + \mathrm{cycl}(F, G, H) = 0\,. \tag{2.11}$$

Proceeding in the same way as in [1, 2], we now evaluate the left-hand side of (2.7) to find a simple local expression for the master equation (2.7) that this generalized model Σ has to satisfy. In order to derive this local expression the following properties of the functional derivatives are needed

$$\begin{aligned} \frac{\overrightarrow{\delta}}{\delta K^*_{P_1}(u,\tau)} K^*_{P_2}(u',\tau') &= \delta^{P_1}_{P_2} \delta^n(u-u')\delta^n(\tau-\tau')\,, \\ \epsilon\left(\frac{\overrightarrow{\delta}}{\delta K^P(u,\tau)}\right) &= \epsilon\left(\frac{\overleftarrow{\delta}}{\delta K^P(u,\tau)}\right) = \epsilon_P + n\,, \\ \epsilon\left(\frac{\overrightarrow{\delta}}{\delta DK^P(u,\tau)}\right) &= \epsilon\left(\frac{\overleftarrow{\delta}}{\delta DK^P(u,\tau)}\right) = \epsilon_P + n + 1\,. \end{aligned} \tag{2.12}$$

A straightforward calculation of the master equation (2.7) now gives

$$\begin{aligned} 0 &= \frac{1}{2}(\Sigma, \Sigma) = \frac{1}{2}\int_{\mathcal{M}} d^n u d^n \tau (\mathcal{L}_n, \mathcal{L}_n)_n \\ &= \int_{\mathcal{M}} d^n u d^n \tau \left(\frac{1}{2}(S, S)_n + D\mathcal{L}_n - D\left(DK^P \frac{\overrightarrow{\partial}}{\partial DK^P} S + DK^*_P \frac{\overrightarrow{\partial}}{\partial DK^*_P} S\right)\right), \end{aligned} \tag{2.13}$$

where we have introduced the following bracket for the local functions $A(u,\tau)$ and $B(u,\tau)$

$$(A,B)_n = \left(A\frac{\overleftarrow{\partial}}{\partial K^P} + (-1)^{\epsilon_P+\epsilon(A)} D\left(A\frac{\overleftarrow{\partial}}{\partial DK^P}\right)\right)\left(\frac{\overrightarrow{\partial}}{\partial K^*_P}B + (-1)^{\epsilon_P+n} D\left(\frac{\overrightarrow{\partial}}{\partial DK^*_P}B\right)\right) - (A\leftrightarrow B)(-1)^{(\epsilon(A)+n+1)(\epsilon(B)+n+1)}. \tag{2.14}$$

In order for the master action Σ to describe a possible consistent gauge field theory (i.e. satisfy the master equation (2.13)), we require the boundary conditions on the superfields to satisfy

$$\int_{\mathcal{M}} d^n u d^n \tau \left(D\mathcal{L}_n(u,\tau) - D\left(DK^P\frac{\overrightarrow{\partial}}{\partial DK^P}S + DK^*_P\frac{\overrightarrow{\partial}}{\partial DK^*_P}S\right)\right) = 0. \tag{2.15}$$

Note that this condition is trivially satisfied for manifolds without a boundary, or for Dirichlet boundary conditions on the fields for manifolds with a boundary. From (2.13) it now follows that we may require the local function S to be a solution to the local master equation

$$(S,S)_n = 0, \tag{2.16}$$

in terms of the bracket (2.14). In principle, more general forms of boundary conditions than (2.15) are possible in which case (2.16) has to be modified.

The bracket (2.14) reduces to the ordinary local bracket if the local functions, such as S, does not depend on terms involving DK^P, DK^*_P. In this case (2.14)–(2.16) are the same as the ones found in [2]. It is the new bracket (2.14) and the introduction of super multiplier fields that makes it possible to derive higher order gauge field theories from the superfield algorithm.

The local expression (2.14) has the graded symmetry property

$$(F,G)_n = -(-1)^{(\epsilon(F)+n+1)(\epsilon(G)+n+1)}(G,F)_n, \tag{2.17}$$

carries (1-n) units of ghost number

$$gh_{\#}(F,G)_n = gh_{\#}F + gh_{\#}G + 1 - n, \tag{2.18}$$

and (n+1) units of parity

$$\epsilon((F,G)_n) = \epsilon(F) + \epsilon(G) + n + 1. \tag{2.19}$$

Due to the D-terms in (2.14) the Jacobi-identity and the Leibniz rule for this local bracket are not easily evaluated, but since the bracket (2.14) is a local expression of the BV-bracket (2.8) the Jacobi-identity and Leibniz rule of (2.14) are valid up to a total derivative, i.e. modulo a D-term [9]. The derivations of the Leibniz rule and the Jacobi identity for higher order theories are considered in great detail in [9] for the bosonic case.

The equations of motion following from the master action (2.1) are given by

$$DK^P = (S,K^P)_n, \qquad DK^*_P = (S,K^*_P)_n. \tag{2.20}$$

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In line with [1, 2], the operator D will now have the role of a BRST-charge operator when acting on the superfield and the associated superfield, since $D^2 = 0$. However, in order to have equations of motion of the form (2.20) for a general function $\Upsilon(K^P, DK^P, K^*_P, DK^*_P)$ we need to introduce a new operator Q replacing D. This operator is given by

$$Q\Upsilon := D\Upsilon + D\left(DK^P \frac{\vec{\partial}}{\partial DK^P}\Upsilon + DK^*_P \frac{\vec{\partial}}{\partial DK^*_P}\Upsilon\right), \tag{2.21}$$

with $\epsilon(Q) = 1$ and $gh_{\#}Q = 1$. Using the equations of motion above we find

$$Q\Upsilon = (S, \Upsilon)_n\,, \tag{2.22}$$

which implies that $QS = (S,S)_n$ and $Q^2S = (S,(S,S)_n)_n$. Note that expression (2.22) above reduces to (2.20) if Υ is just a superfield or an associated superfield. Q is a nilpotent operator using (2.20), i.e. $Q^2 = 0$, which gives Q an interpretation of a BRST-charge operator. Since $0 = Q^2\Upsilon = (S,(S,\Upsilon)_n)_n$ it follows from the graded Jacobi identity that consistency requires that $(S,S)_n = 0$. This argument supports our proposal (2.16). Note that with the operator Q we may write the conditions (2.15) as

$$\int_{\mathcal{M}} d^nud^n\tau\,(D(K^*_P DK^P) - QS) = 0\,, \tag{2.23}$$

where the last term disappear on-shell. For manifolds without boundary we can disregard any D-exact terms. For reasonable choices of boundary conditions on the fields, such as for example Dirichlet conditions, the D-exact terms vanishes on the boundary. All of the analysis made in the rest of this paper, assumes that such a choice has been made.

The expansion of the superfields K^P and K^*_P in terms of the odd coordinates τ^a lead to component fields which are either conventional BV fields or -antifields. Since the original fields constitute the ghost number zero components of the superfields K^P and K^*_P, one obtains the following rules [2] for extracting the n-dimensional classical field theory corresponding to a given master action Σ of the form (2.1) and (2.2):

$$\begin{aligned} d^nud^n\tau &\to 1 \\ D &\to \text{exterior derivative } d \\ K^P : gh_{\#}K^P = k \geq 0 &\to \text{k-form field } k^P \text{where,} \\ &\quad\ \epsilon(k^P) = \epsilon_P + k \\ K^*_P : gh_{\#}K^*_P = (n-1-k) \geq 0 &\to (n\text{-}1\text{-}k) - \text{form field } k^*_P \text{ where,} \\ &\quad\ \epsilon(k^*_P) = \epsilon_P + k \\ \text{all other superfields} &\to 0 \\ \text{pointwise multiplication} &\to \text{wedge product}\,. \end{aligned} \tag{2.24}$$

The variation of the superfields in Σ are given by

$$\delta_\Sigma K^P := (\Sigma, K^P) = (-1)^n(DK^{P_1} - (S, K^P)_n)\,, \tag{2.25}$$

$$\delta_\Sigma K^*_P := (\Sigma, K^*_P) = (-1)^n(DK^*_{P_1} - (S, K^*_P)_n)\,, \tag{2.26}$$

from which one can determine the gauge transformations of a classical model by applying the rules (2.24) in the same way as was considered in [6].

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3. Generalized superfield algorithm

We begin this section by considering some of the limitations of the superfield algorithm in [2] when it comes to describing higher order gauge field theories. Then we show how the introduction of non-dynamical super multiplier fields Λ^P and Λ^*_P can resolve these limitations.

Consider the reduction rules (2.24) from which the original model of a given master action may be derived. Now, in general a master action in which all the fields are dynamical, does not provide for the possibility of an explicit relation between the fields and their associated fields. However, a master action with only dynamical fields might provide for an implicit relation between the fields and the associated fields. This is for instance the case for theories like topological Yang-Mills (TYM), in which the associated field can be eliminated by means of its own equation of motion. This implies that the original model, i.e. classical TYM only contain one field, namely the gauge connection 1-form, exactly as it should. It is easy to see that such an implicit relation between dynamical fields and associated fields is only possible if there exists a quadratic term of either the field or the associated field in the master action. Only then may the equation of motion for that field give a sensible relation to its corresponding associated field. However, a quadratic term is only possible in even dimensions, which rules out such implicit relations for all models in odd dimensions. Moreover, for many models in even dimensions such 'searched for' relations are not possible either. The reason for this is that the ghost number prescription of the master action requires the ghost number of the field in the quadratic term to be equal the dimension of the model divided by two, which is not the case for most fields. Important odd dimensional models which cannot be formulated without multiplier fields, includes for example $d = 3$ Chern-Simons models and higher dimensional analogues — all of which clearly contain only one field: the gauge connection 1-form.

Introducing non-dynamical fields into the model can be used to impose (via their equations of motions) relations (constraints) between fields and the associated fields in the master action. Relations which are carried over to the original model, via the rules (2.24). Thus, a typical multiplier term in the master action looks like

$$\Lambda^P \phi_P(K, K^*, DK, DK^*)\,, \tag{3.1}$$

and imposes the relation $\phi_P = 0$ between K and K^* through the equations of motion. Consider now the equations of motion (2.20) following from a given local master action S,

$$DK = (S, K)_n\,. \tag{3.2}$$

Above, the indices are suppressed and K denotes either a superfield or an associated superfield. Since S is linear in the multiplier fields, two things can happen: 1) S does not contain the associated field to a given multiplier field, say Λ, in which case we have $(S, \Lambda) = 0$. 2) S contains the associated field to a given field, say Λ, in which case we have $(S, \Lambda) = \phi$ for some constraint ϕ. On the constraint surface, however, we have also in this case $(S, \Lambda) = 0$. Due to the general form of the equations of motion (2.20) we therefore

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define the super multiplier fields to be those satisfying the following properties,

$$
\begin{aligned}
D\Lambda^P &= 0\,,\\
D\Lambda^*_P &= 0\,.
\end{aligned}
\tag{3.3}
$$

For higher order theories in general, the local master equation (2.16) will include terms containing the de Rham differential D on some fields. If these terms are D-exact they may be ignored — since they may be absorbed by redefinitions of the local bracket $(\ ,\ )_n$ (this might affect the boundary conditions). If not, we must use the equations of motion (2.20) of the corresponding fields in order to obtain terms which are polynomial in the fields only. Depending on the theories considered, we might be forced to use this procedure repeatedly in order to bring all the terms in the local master equation on polynomial form in the fields. Only then may we identify the equations that our coefficients in the master action must satisfy. It is in this sense, that the solutions found to the local master equation are on-shell solutions. The relations imposed by the constraints should also be imposed at this level. This is so since the multiplier fields impose the constraints by means of their equations of motion and we are interested in the equations that the master equation imply for the independent dynamical fields.

The steps above may be summarized:

1) Calculate the local master equation $(S, S)_n = 0$.

2) Impose the constraints by using the equations of motion for the Lagrange super multiplier fields.

3) D-exact terms in the master equation may be set to zero since they do not contribute to the solution.

4) Eliminate all kinetic terms that are left by using the appropriate equations of motion for the corresponding fields (with kinetic terms is here meant terms containing D on some fields).

The main motivation for this algorithm is the fact that it seems to work on all models tested so far. In some cases, for example topological Yang-Mills in $n = 4$ for which we have alternative formulations, the algorithm above generates identical solutions to the master equation as those obtained by considering only dynamical fields.

4. Topological Yang-Mills theory

Topological Yang-Mills theory in four dimensions, originally considered in [10, 11], is defined by the action

$$
S_{TYM} = \int \mathrm{tr}[F\wedge F] = \int \mathrm{tr}[dA^2 + 2A^2 dA + A^4]\,. \tag{4.1}
$$

The field strength F is related to the gauge connection 1-form A as $F = dA + A^2$. Locally, $A = g_a A^a = g_a A^a_\mu dx^\mu$, where g_a satisfies the Lie-algebra $[g_a, g_b] = f_{ab}{}^c g_c$. The expression above can now be written in terms of a symmetric group metric $g_{ab} := \mathrm{tr}[g_a g_b]$ and a totally

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Superfield	Ghost number	Grassmann parity
T^*_E	2	0
T^D	1	1
Λ^*_D	0	1
Λ^D	2	0

Table 1: Superfield characteristics of TYM.

antisymmetric structure coefficient $f_{abc} = f_{ab}{}^d g_{cd}$,

$$S_{TYM} = \int g_{ab} dA^a dA^b + f_{ab}{}^d g_{dc} A^a A^b dA^c \,. \tag{4.2}$$

This theory can be generated from a superfield formulation by introducing a set of superfields with the characteristics in table 1.

The Grassmann parities of the superfields are chosen such that they are even after reduction to the original theory, according to the rules in (2.24). The master action we consider is given by

$$\Sigma_{TYM} = \int_{\mathcal{M}} d^4u d^4\tau (-T^*_E DT^E - S_4) \,, \tag{4.3}$$

where the local master action is

$$S_4 = T^*_{E_1} \omega^{E_1}{}_{E_2E_3} T^{E_2} T^{E_3} + \omega_{E_1E_2E_3E_4} T^{E_1} T^{E_2} T^{E_3} T^{E_4} + \Lambda^{E_1} (T^*_{E_1} - g_{E_1E_2} DT^{E_2}) \,. \tag{4.4}$$

Above, (u^a, τ^a) denotes a set of Grassmann even and odd local coordinates on a supermanifold $\mathcal{M}$. Since S_4 is even it follows from the Grassmann parities of the superfields that the coefficients $\omega^{E_1}{}_{E_2E_3}$ and $g_{E_1E_2}$ are even. For simplicity, we choose these coefficients to be independent of (u, τ). As will be seen, this restriction is natural for the limit (2.24) of (4.3) to be a topological Yang-Mills theory.

The constraint relating the associated superfield T^*_E with the superfields T^E is given by

$$\phi_{E_1} = T^*_{E_1} - g_{E_1E_2} DT^{E_2} \,, \tag{4.5}$$

and is first-class with respect to $(\ ,\ )_4$, since

$$(\phi_{E_1}, \phi_{E_2})_4 = 2D(g_{E_1E_2}) = 0 \,. \tag{4.6}$$

The constraint is needed in order to reproduce the topological Yang-Mills action (4.2) from the master action (4.3), after taking the limit (2.24).

The next step is to solve the local master equation (2.16) for this model. This is done at the level of the equations of motions. In other words, we have to implement (2.20) to obtain a solution of the local master equation.

The procedure goes as follows; first the local master equation is calculated, thereafter the constraint $T^*_{E_1} = g_{E_1E_2} DT^{E_2}$ is imposed and finally the DT^E-terms are replaced by their equations of motion $DT^E = -\omega^E{}_{E_1E_2} T^{E_1} T^{E_2} - \Lambda^E$. By performing the procedure, the

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master equation imposes the following set of equations:

$$
\begin{aligned}
0 &= \left(g_{E_5E_6}\omega^{E_6}{}_{E_1E_2}\omega^{E_5}{}_{E_3E_4} + g_{E_6E_4}\omega^{E_6}{}_{E_1E_5}\omega^{E_5}{}_{E_2E_3} - 2\omega_{E_1E_2E_3E_4}\right) T^{E_1}T^{E_2}T^{E_3}\Lambda^{E_4}\,, \\
0 &= g_{EE_1}\omega^{E}{}_{E_3E_2}\Lambda^{E_1}\Lambda^{E_2}T^{E_3}\,, \\
0 &= (2\omega_{E_1E_2E_3E}\omega^{E}{}_{E_4E_5} - g_{E_6E_7}\omega^{E_7}{}_{E_1E_2}\omega^{E_6}{}_{E_3E}\omega^{E}{}_{E_4E_5})T^{E_1}T^{E_2}T^{E_3}T^{E_4}T^{E_5}\,.
\end{aligned}
$$

We observe that if we choose the coefficient $g_{E_1E_2}$ to be symmetric and invertible (with inverse $g^{E_1E_2}$) we may write the equations above as

$$
\begin{aligned}
\omega_{(E_1E_2E}\omega^{E}{}_{E_3)E_4} &= 0\,, && (4.7)\\
\omega_{E_1E_2E_3} &= -\omega_{E_3E_2E_1}\,, && (4.8)\\
\omega_{E_1E_2E_3E_4} &= \frac{1}{2}\omega_{E_1E_2E}\omega^{E}{}_{E_3E_4}\,. && (4.9)
\end{aligned}
$$

The braces indicates a graded symmetrization over the enclosed indices. Equation (4.7) then imposes the Jacobi-identity on the coefficients $\omega^{E_1}{}_{E_2E_3}$, such that these may be interpreted as the structure coefficients of some Lie-algebra. $g_{E_1E_2}$ is in this case naturally identified as the group metric of that Lie-algebra. Since the fields T^E are Grassmann odd, we may assume that $\omega^{E_1}{}_{E_2E_3} = -\omega^{E_1}{}_{E_3E_2}$ which together with equation (4.8) implies the total antisymmetry of the coefficients $\omega_{E_1E_2E_3} = g_{E_1E_4}\omega^{E_4}{}_{E_2E_3}$. Thus, the master equation implies that $\omega^{E_1}{}_{E_2E_3}$ are structure coefficients of a semi-simple Lie-algebra in this case. This is to be expected from consistency requirements, since we have assumed $g_{E_1E_2}$ to be an invertible group metric. Equation (4.9) gives the solution for $\omega_{E_1E_2E_3E_4}$. Due to the symmetrization over the T^E-fields in the master action, this term drops out as a consequence of the Jacobi-identity. This is also the case on the classical level for $\int \mathrm{tr}[F\wedge F]$, since the cyclicity of the trace and antisymmetry of the wedge product forces the A^4-term to vanish.

The solved for master action can now be written,

$$
\Sigma_{TYM} = -\int_{\mathcal{M}} d^4u d^4\tau (T^*_E DT^E + T^*_{E_1}\omega^{E_1}{}_{E_2E_3}T^{E_2}T^{E_3} + \Lambda^{E_1}(T^*_{E_1} - g_{E_1E_2}DT^{E_2})\,. \qquad (4.10)
$$

Taking the limit (2.24) reduces T^*_E to an even two form t^*_E, T^E to an even one form t^E and Λ^E to an even two form λ^E. By implementing the constraint

$$
t^*_{E_1} = g_{E_1E_2}dt^{E_2}\,, \qquad (4.11)
$$

the following classical action is obtained

$$
\Sigma_{Cl.} = -\int g_{E_1E_2}dt^{E_1}\wedge dt^{E_2} + g_{E_1E}\omega^{E}{}_{E_2E_3}dt^{E_1}\wedge t^{E_2}\wedge t^{E_3}\,, \qquad (4.12)
$$

where the coefficients $\omega_{E_1E_2E_3}$ belongs to a semi-simple Lie-algebra. This is exactly the topological Yang-Mills action (4.2) if we identify $t^E \to A^a$, $\omega_{E_1E_2E_3} \to f_{abc}$ and $g_{E_1E_2} \to -g_{ab}$. Hence, a topological Yang-Mills theory has been generated using a generalized superfield algorithm.

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As previously discussed, the gauge transformations of the original model can be calculated using the superfield formalism. The Σ-variation of the superfield T^E is

$$\delta_\Sigma T^E = DT^E - (\omega^E{}_{E_1E_2}T^{E_1}T^{E_2} + \Lambda^E)\,. \tag{4.13}$$

Applying the reduction rules (2.24), the gauge transformations can be derived

$$\delta t^E = d\tilde{t}^E - 2\omega^E{}_{E_1E_2}\tilde{t}^{E_1}t^{E_2} - \tilde{\lambda}^E\,. \tag{4.14}$$

The tilde quantities $\tilde{t}^E, \tilde{\lambda}^E$ above are $0,1$ -form gauge parameters respectively. The first two terms in (4.14) correspond to the covariant Yang-Mills derivative and the last one to the topological shift symmetry.

The topological Yang-Mills theory can also be written in a superfield formulation using auxiliary[1] fields instead of multiplier fields. In this case the local master action S is in comparison with (4.4) written as

$$S' = T^*_{E_1}T^*_{E_2}\omega^{E_1E_2} + T^*_{E_1}\omega^{E_1}{}_{E_2E_3}T^{E_2}T^{E_3}\,, \tag{4.15}$$

where the superfield T^E and the auxiliary associated superfield T^*_E have the same properties as above and the coefficients are both even.

5. $n = 5$ Chern-Simons theory

The classical action for five dimensional (pure) Chern-Simons theory [12, 13] is defined by the lagrangian density

$$\mathcal{L}_{CS5} = \mathrm{tr}\left[dA^2A + \frac{3}{2}dAA^3 + \frac{3}{5}A^5\right], \tag{5.1}$$

where $A = g_aA^a$ denotes the usual Lie-algebra valued 1-form (gauge connection). Locally, the gauge connection is given by $A = g_aA^a_\mu dx^\mu$ where the generators g_a satisfies a Lie-algebra $[g_a, g_b] = f_{ab}{}^cg_c$. If we write the density (5.1) explicitly in terms of the structure constants of the Lie-algebra, we get

$$\mathcal{L}_{CS5} = g_{abc}dA^adA^bA^c + \frac{3}{4}f_{cd}{}^eg_{abe}dA^aA^bA^cA^d + \frac{3}{20}f_{ab}{}^ff_{cd}{}^hg_{fhe}A^aA^bA^cA^dA^e\,. \tag{5.2}$$

Above, the rank 3-tensor g_{abc} is given by $g_{abc} := \mathrm{tr}[g_ag_bg_c]$ and it is obviously invariant under the adjoint action of the gauge group. In general, for Chern-Simons theories in $(2n+1)$ dimensions, $g_{a_1\cdots a_{n+1}}$ is assumed to be a rank $n+1$ symmetric tensor. The reason for this assumption stems from the fact that,

$$d\mathcal{L}^{2n+1}_{CS} = g_{a_1\cdots a_{n+1}}F^{a_1}\wedge\cdots\wedge F^{a_{n+1}}\,, \tag{5.3}$$

where the 2-form Yang-Mills field strength F^a, projects out the symmetric part of $g_{a_1\cdots a_{n+1}}$. However, the symmetry of $g_{a_1\cdots a_{n+1}}$ does not follow from its definition, as for example can be seen from the definition of g_{abc} above. In [8, 7] it is investigated how the algebraic properties

[1] We define auxiliary fields as dynamical fields that can be eliminated by means of their own equations of motion. In our formulation the super multiplier fields are not dynamical.

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Superfield	Ghost number	$\mathbb{Z}_2 - grading$
S^D	1	1
S^*_D	3	1
Λ^D	2	0
Λ^*_D	2	0

Table 2: Superfield characteristics of $n = 5$, CS-theory.

of $g_{a_1\cdots a_{n+1}}$ determines the exact nature of the constraints in the action (5.1) and the higher dimensional versions thereof. One of the conclusions of their investigation is that higher dimensional ($n \geq 5$) Chern-Simons theories generically do possess local degrees of freedom and are therefore not topological gauge theories. Below, we will see how the tensor g_{abc} naturally enters in the superfield formulation and what kind of algebraic constraints that are imposed on it as a consequence of the master equation.

Now, a superfield formulation of the five dimensional Chern-Simons theory may be obtained by introducing superfields S^D, S^*_D, Λ^D and Λ^*_D with the characteristics in table 2.

The specific choice of the parities of the superfields above, make sure that the fields in the original model are Grassmann even after the limit (2.24) is taken.

The master action we consider is then given by

$$\Sigma_{CS5} = \int_{\mathcal{M}} d^5u d^5\tau \mathcal{L}_5 = \int_{\mathcal{M}} d^5u d^5\tau (S^*_D DS^D - S_5)\,, \tag{5.4}$$

where the local master action has the form

$$\begin{aligned} -S_5 = {} & \frac{3}{5}\omega_{D_1D_2D_3D_4D_5}S^{D_1}S^{D_2}S^{D_3}S^{D_4}S^{D_5} + \frac{3}{2}S^*_{D_1}\omega^{D_1}{}_{D_2D_3}S^{D_2}S^{D_3} + \\ & + \Lambda^{D_1}(S^*_{D_1} - g_{D_1D_2D_3}DS^{D_2}S^{D_3})\,. \end{aligned} \tag{5.5}$$

It follows from (2.4) and (2.5) that the coefficients $\omega_{D_1D_2D_3D_4D_5}$, $\omega^{D_1}{}_{D_2D_3}$, $g_{D_1D_2D_3}$ are all even and have ghost number zero. The coefficients are chosen to be independent of the coordinates (u, τ) and it is obvious from (5.5) which symmetries they possess with respect to their indices. The constraint that relates the field S^D with the associated field S^*_D is given by $\phi_{D_1} = 0$, where

$$\phi_{D_1} = (S^*_{D_1} - g_{D_1D_2D_3}DS^{D_2}S^{D_3})\,. \tag{5.6}$$

This choice of the constraint implies that a possible limit (2.24) of the master action (5.4) is the classical five dimensional Chern-Simons action (5.1). This is seen by solving the constraint in (5.5) and observing that we then obtain the same polynomiality in the fields as in the classical action. It should be pointed out, however, that there in general exist other possible choices of the constraint. We observe that since

$$(\phi_D, \phi_{D'})_5 = -2D(g_{DD_1D'}S^{D_1} + g_{DD'D_1}S^{D_1}) \tag{5.7}$$

is D-exact, ϕ_{D_1} is first-class with respect to $(\ ,\ )_5$.

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Let us now consider the (local) master equation for the action (5.5). Following the procedure of the generalized superfield algorithm we calculate $(S,S)_5 = 0$ and then use that

$$\begin{aligned} S^*_{D_1} &= g_{D_1 D_2 D_3} D S^{D_2} S^{D_3} \,, \\ D S^D &= -\frac{3}{2}\omega^D{}_{D_1 D_2} S^{D_1} S^{D_2} - \Lambda^D \,. \end{aligned} \tag{5.8}$$

By collecting terms with identical field structure in the master equation we obtain three equations,

$$\begin{aligned} 0 &= \omega_{D_1 D_2 D_3 D_4 D}\, \omega^D{}_{D_5 D_6} S^{D_1} S^{D_2} S^{D_3} S^{D_4} S^{D_5} S^{D_6} - \\ &\quad - \frac{3}{2} g_{D_7 D_8 D_1} \omega^{D_8}{}_{D_2 D_3} \omega^{D_7}{}_{D_4 D_9} \omega^{D_9}{}_{D_5 D_6} S^{D_1} S^{D_2} S^{D_3} S^{D_4} S^{D_5} S^{D_6} \,, \\ 0 &= \frac{1}{2}(g_{D_1 D_2 D}\omega^D{}_{D_3 D_4} + g_{D_1 D D_2}\omega^D{}_{D_3 D_4}) \Lambda^{D_1} \Lambda^{D_2} S^{D_3} S^{D_4} - \\ &\quad - g_{D_5 D_1 D_3} \omega^{D_5}{}_{D_4 D_2} \Lambda^{D_1} \Lambda^{D_2} S^{D_3} S^{D_4} \,, \\ 0 &= \omega_{D_1 D_2 D_3 D_4 D} \Lambda^D S^{D_1} S^{D_2} S^{D_3} S^{D_4} - \\ &\quad - \frac{3}{2} g_{D_5 D D_1} \omega^{D_5}{}_{D_2 D_6} \omega^{D_6}{}_{D_3 D_4} \Lambda^D S^{D_1} S^{D_2} S^{D_3} S^{D_4} - \\ &\quad - \frac{3}{2} g_{D_5 D_6 D_1} \omega^{D_6}{}_{D_2 D_3} \omega^{D_5}{}_{D_4 D} \Lambda^D S^{D_1} S^{D_2} S^{D_3} S^{D_4} + \\ &\quad + \frac{3}{2} g_{D D_5 D_6} \omega^{D_6}{}_{D_1 D_2} \omega^{D_5}{}_{D_3 D_4} \Lambda^D S^{D_1} S^{D_2} S^{D_3} S^{D_4} \,. \end{aligned} \tag{5.9}$$

Note that the polynomials in the fields above act as projection operators, projecting out terms with certain symmetries. For example, an equation like

$$0 = \gamma_{D_1 D_2 D_3 D_4 D_5 D_6} S^{D_1} S^{D_2} S^{D_3} S^{D_4} S^{D_5} S^{D_6}$$

only contains information about the totally antisymmetric part of the coefficient $\gamma_{D_1 D_2 D_3 D_4 D_5 D_6}$. We now look for the existence of solutions to the equations (5.9) in which the coefficients $\omega^{D_1}{}_{D_2 D_3}$ belong to a semi-simple Lie-algebra. Since we know what classical limit we would like to have from the master action, it is clear that these are the interesting solutions from a Chern-Simons point of view. This means in particular that the set of $\omega^{D_1}{}_{D_2 D_3}$ will satisfy the Jacobi-identity. Thus, we look for solutions that satisfies,

$$\omega^{D_5}{}_{(D_1|D_4} \omega^{D_4}{}_{|D_2 D_3)} = 0 \,. \tag{5.10}$$

In the equation above the braces around the indices stand for graded symmetrization. The Jacobi-identity implies that the system (5.9) reduces to,

$$0 = \omega_{D_1 D_2 D_3 D_4 D}\, \omega^D{}_{D_5 D_6} S^{D_1} S^{D_2} S^{D_3} S^{D_4} S^{D_5} S^{D_6} \,, \tag{5.11}$$

$$\begin{aligned} 0 &= \frac{1}{2}(g_{D_1 D_2 D}\omega^D{}_{D_3 D_4} + g_{D_1 D D_2}\omega^D{}_{D_3 D_4}) \Lambda^{D_1} \Lambda^{D_2} S^{D_3} S^{D_4} - \\ &\quad - g_{D_5 D_1 D_3} \omega^{D_5}{}_{D_4 D_2} \Lambda^{D_1} \Lambda^{D_2} S^{D_3} S^{D_4} \,, \end{aligned} \tag{5.12}$$

$$\begin{aligned} 0 &= \omega_{D_1 D_2 D_3 D_4 D} \Lambda^D S^{D_1} S^{D_2} S^{D_3} S^{D_4} - \\ &\quad - \frac{3}{2} g_{D_5 D_6 D_1} \omega^{D_6}{}_{D_2 D_3} \omega^{D_5}{}_{D_4 D} \Lambda^D S^{D_1} S^{D_2} S^{D_3} S^{D_4} + \\ &\quad + \frac{3}{2} g_{D D_5 D_6} \omega^{D_6}{}_{D_1 D_2} \omega^{D_5}{}_{D_3 D_4} \Lambda^D S^{D_1} S^{D_2} S^{D_3} S^{D_4} \,. \end{aligned} \tag{5.13}$$

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Note that the second equation above gives an algebraic relation between the coefficients in the constraints and the structure coefficients of the Lie-algebra. Thus, the master equation enforces a relation between the two a priori unrelated sets of objects $g_{D_1D_2D_3}$ and $\omega^{D_1}{}_{D_2D_3}$. The third equation gives a solution for the coefficient $\omega_{D_1D_2D_3D_4D_5}$ in terms of the Lie-algebra coefficients and the coefficients in the constraints. The first equation can then be interpreted as a self-consistency condition of that solution.

Previously it was noted that the polynomials of the fields in the master equation act as projectors and in general one might add tensors to the solutions that vanishes under these projections. We are however, looking for the simplest possible solutions and therefore we do not consider this issue. Further, it is reasonable to believe that these additional tensors in many cases can be made to vanish by suitable redefinitions of the fields.

We will now study the solution of the equations (5.11)–(5.13) in more detail. Consider equation (5.12), it implies that

$$g_{DD_1D_2}\omega^D{}_{D_3D_4} = \frac{1}{2}(g_{D_1D_4D}\omega^D{}_{D_2D_3} + g_{D_1DD_4}\omega^D{}_{D_2D_3})\,. \tag{5.14}$$

Inserted into (5.13) we get,

$$\begin{aligned}
0 = {} & \omega_{D_1D_2D_3D_4D}\Lambda^D S^{D_1}S^{D_2}S^{D_3}S^{D_4} - \\
& - \frac{3}{2}g_{D_5D_6D_1}\omega^{D_6}{}_{D_2D_3}\omega^{D_5}{}_{D_4D}\Lambda^D S^{D_1}S^{D_2}S^{D_3}S^{D_4} - \\
& - \frac{3}{2}g_{DD_6D_5}\omega^{D_6}{}_{D_1D_2}\omega^{D_5}{}_{D_3D_4}\Lambda^D S^{D_1}S^{D_2}S^{D_3}S^{D_4} + \\
& + 3g_{D_6DD_1}\omega^{D_6}{}_{D_2D_5}\omega^{D_5}{}_{D_3D_4}\Lambda^D S^{D_1}S^{D_2}S^{D_3}S^{D_4}\,.
\end{aligned}$$

The last term in the system above vanishes as a consequence of the Jacobi-identity, since the subscript indices D_2, D_3 and D_4 are symmetrized over. This gives the solution for $\omega_{D_1D_2D_3D_4D}$,

$$\omega_{D_1D_2D_3D_4D} = \frac{3}{2}\left(g_{D_5D_6D_1}\omega^{D_6}{}_{D_2D_3}\omega^{D_5}{}_{D_4D} + g_{DD_6D_5}\omega^{D_6}{}_{D_1D_2}\omega^{D_5}{}_{D_3D_4}\right)\,. \tag{5.15}$$

The solution (5.15) is consistent with equation (5.11) since

$$\begin{aligned}
& \omega_{D_1D_2D_3D_4D}\omega^D{}_{D_7D_8}S^{D_1}S^{D_2}S^{D_3}S^{D_4}S^{D_7}S^{D_8} = \\
& = \frac{3}{2}g_{DD_6D_5}\omega^{D_6}{}_{D_1D_2}\omega^{D_5}{}_{D_3D_4}\omega^D{}_{D_7D_8}S^{D_1}S^{D_2}S^{D_3}S^{D_4}S^{D_7}S^{D_8} \\
& = \frac{3}{4}g_{D_6D_8D}\omega^D{}_{D_5D_7}\omega^{D_6}{}_{D_1D_2}\omega^{D_5}{}_{D_3D_4}S^{D_1}S^{D_2}S^{D_3}S^{D_4}S^{D_7}S^{D_8} + \\
& \quad + \frac{3}{4}g_{D_6DD_8}\omega^D{}_{D_5D_7}\omega^{D_6}{}_{D_1D_2}\omega^{D_5}{}_{D_3D_4}S^{D_1}S^{D_2}S^{D_3}S^{D_4}S^{D_7}S^{D_8} \\
& = 0
\end{aligned}$$

as a consequence of the Jacobi-identity. A superfield formulation of five dimensional Chern-Simons theories is thus given by the action

$$\Sigma = \int_{\mathcal{M}} d^5u d^5\tau\, \mathcal{L}_5\,, \tag{5.16}$$

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where

$$\mathcal{L}_5 = S^*_{D_1} DS^{D_1} + \frac{9}{10}\left(g_{D_7 D_6 D_1} + g_{D_1 D_7 D_6}\right) \omega^{D_6}{}_{D_2 D_3} \omega^{D_7}{}_{D_4 D_5} S^{D_1} S^{D_2} S^{D_3} S^{D_4} S^{D_5} + \\ + \frac{3}{2} S^*_{D_1} \omega^{D_1}{}_{D_2 D_3} S^{D_2} S^{D_3} + \Lambda^{D_1}(S^*_{D_1} - g_{D_1 D_2 D_3} DS^{D_2} S^{D_3}) \,. \tag{5.17}$$

After eliminating the super multiplier field Λ, using relation (5.14) and performing a partial integration of the kinetic term, we get

$$\mathcal{L}_5|_\Lambda = \frac{1}{2}\left(g_{D_1 D_2 D_3} + g_{D_2 D_3 D_1}\right) DS^{D_2} S^{D_3} DS^{D_1} + \\ + \frac{9}{10}\left(g_{D_7 D_6 D_1} + g_{D_1 D_7 D_6}\right) \omega^{D_6}{}_{D_2 D_3} \omega^{D_7}{}_{D_4 D_5} S^{D_1} S^{D_2} S^{D_3} S^{D_4} S^{D_5} + \\ + \frac{3}{4}\left(g_{D_2 D_5 D_1} + g_{D_2 D_1 D_5}\right) \omega^{D_1}{}_{D_3 D_4} DS^{D_4} S^{D_5} S^{D_2} S^{D_3} \,.$$

In order to obtain Chern-Simons theory we impose symmetry between the first two indices of the g tensor,

$$g_{D_1 D_2 D_3} = g_{D_2 D_1 D_3} \,. \tag{5.18}$$

This choice is very natural since if we define

$$\tilde{g}_{D_1 D_2 D_3} = g_{D_1 D_2 D_3} + g_{D_1 D_3 D_2} \,, \tag{5.19}$$

$\tilde{g}_{D_1 D_2 D_3}$ will be totally symmetric. We get thus,

$$\mathcal{L}_5|_\Lambda = \frac{1}{2}\tilde{g}_{D_1 D_2 D_3} DS^{D_2} S^{D_3} DS^{D_1} + \frac{9}{10}\tilde{g}_{D_7 D_6 D_1} \omega^{D_6}{}_{D_2 D_3} \omega^{D_7}{}_{D_4 D_5} S^{D_1} S^{D_2} S^{D_3} S^{D_4} S^{D_5} + \\ + \frac{3}{4}\tilde{g}_{D_2 D_5 D_1} \omega^{D_1}{}_{D_3 D_4} DS^{D_4} S^{D_5} S^{D_2} S^{D_3} \,. \tag{5.20}$$

Note that this expression has exactly the same structure as the one in (5.2). Thus, we expect this to be the master action of the five dimensional Chern-Simons theory by means of an appropriate redefinition of the coefficients. In the case of an abelian Chern-Simons theory this is obtained from (5.20) by applying the reduction rules (2.24) and identifying $S^{D_1} \to A^a, \omega^{D_1}{}_{D_2 D_3} \to 0, \frac{1}{2}\tilde{g}_{D_1 D_2 D_3} \to g_{abc}$.

6. $n = 3$ theories

In this section we take a look at some three dimensional models. When trying to formulate $n = 3$ Chern-Simons theories in terms of a master action, complications occur when one uses the superfield algorithm. The reason for this is that the most natural (simplest) constraints that relates superfields and associated superfields are generically second class with respect to $(\ ,\ )_3$ in this case. This implies that we obtain some restrictions on the Lagrange multipliers Λ^F that enforces the constraints. Consider the classical action of $d = 3$ Chern-Simons,

$$\mathcal{L}_{CS3} = \mathrm{tr}\left[AdA + \frac{2}{3}A^3\right] . \tag{6.1}$$

The notation is identical to that of the preceding section. The characteristics of the fields are in table 3.

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Superfield	Ghost number	$\mathbb{Z}_2 - grading$
U^F	1	1
U^*_F	1	1
Λ^F	2	0
Λ^*_F	0	0

Table 3: Superfield characteristics of $n = 3$, CS-theory.

The specific parities above are chosen as to produce even fields in the limit (2.24) for the following master action:

$$\Sigma = \int_{\mathcal{M}} d^3u d^3\tau (U^*_F DU^F - U^*_{F_1}\omega^{F_1}{}_{F_2F_3}U^{F_2}U^{F_3} - \Lambda^{F_1}(U^*_{F_1} - g_{F_1F_2}U^{F_2}))\,. \tag{6.2}$$

The constraints above are the simplest ones possible if we want the preceding master action to reduce to an action defined by (6.1). The constraints ϕ^F are given by

$$\phi_{F_1} = U^*_{F_1} - g_{F_1F_2}U^{F_2}\,, \tag{6.3}$$

and their algebra with respect to (,) is given by,

$$(\phi^{F_1}, \phi^{F_2})_3 = -g^{F_1F_2} - g^{F_2F_1}\,. \tag{6.4}$$

This implies that ϕ^{F_1} is of first-class with respect to $(\, , \,)_3$ only if $g^{F_1F_2}$ is antisymmetric. We cannot identify $g_{F_1F_2}$ with some group metric, since in that case it must be symmetric. Without making any assumptions about the symmetries of the $g_{F_1F_2}$ coefficient, we now calculate the master equation,

$$\begin{aligned} 0 = (S,S)_3 \;=\; & 2g_{F_1F_5}U^{F_5}\omega^{F_1}{}_{F_2F}\omega^{F}{}_{F_3F_4}U^{F_2F_3F_4} + 2g_{F_1F_5}\omega^{F_1}{}_{F_2F}U^{F_5}U^{F_2}\Lambda^F - \\ & - \Lambda^{F_1}g_{F_1F}\omega^{F}{}_{F_3F_4}U^{F_3}U^{F_4} - \Lambda^{F_1}g_{F_1F}\Lambda^F\,, \end{aligned} \tag{6.5}$$

which gives the following three equations,

$$\begin{aligned} 0 &= g_{F_1F_2} + g_{F_2F_1}\,, \\ 0 &= \omega_{(F_5F_2|F}\omega^{F}{}_{|F_3F_4)}\,, \\ 0 &= \omega_{F_5F_2F} - \omega_{F_2F_5F} - \omega_{FF_5F_2}\,. \end{aligned} \tag{6.6}$$

Note that in the first equation above, the first-class requirement is imposed as a condition via the master equation. Thus, if the model described by (6.2) is to describe a consistent gauge theory, $g_{F_1F_2}$ has to be antisymmetric. The coefficients $\omega_{F_5F_2F} = g_{F_5F_6}\omega^{F_6}{}_{F_2F}$ cannot belong to a semi-simple Lie-algebra in this case and it is precisely that fact that makes the last equation above non-trivial (otherwise we would have $\omega_{F_1F_2F_3} = 0$). The second equation can be viewed as a generalized Jacobi-identity. The classical limit of the theory under consideration is, however in the first-class scenario, given by the non-dynamical model,

$$\int \omega_{F_1F_2F_3}u^{F_1} \wedge u^{F_2} \wedge u^{F_3}\,. \tag{6.7}$$

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The reason for this is that if $g_{F_1F_2}$ is anti-symmetric, we have (for suitable boundary conditions on the fields u^F)

$$\int g_{F_1F_2}u^{F_1}\wedge du^{F2} = -\int g_{F_2F_1}u^{F_2}\wedge du^{F_1}\,. \tag{6.8}$$

The action (6.7) is gauge invariant as a consequence of equation (6.6) since,

$$\delta(\omega_{F_1F_2F_3}u^{F_1}\wedge u^{F_2}\wedge u^{F_3}\wedge) = (\omega_{F_1F_2F_3}-\omega_{F_2F_1F_3}+\omega_{F_3F_2F_1})\delta u^{F_1}\wedge u^{F_2}\wedge u^{F_3}\,. \tag{6.9}$$

There exists also other solutions to this model, when the first-class requirement with anti-symmetric $g_{F_1F_2}$ is not satisfied. From (6.5) we see that we always have the trivial solution $\Lambda^F=0$, which amounts to a three dimensional model with a Lie-algebra symmetry. Another, more interesting possibility is to consider the master equation as imposing constraints on the super multiplier fields Λ^F. It is then easy to see directly from (6.5) that if we set

$$\Lambda^F = \omega^F{}_{F_1F_2}U^{F_1}U^{F_2} \tag{6.10}$$

we have a consistent solution to the master equation, provided the coefficients $\omega^F{}_{F_3F_4}$ satisfies the Jacobi-identity. In this case, the coefficient $g_{F_1F_2}$ may have a group metric interpretation. The classical limit of such a model is given by

$$\int u_F^*\wedge du^F + 2\,\omega^F{}_{F_3F_4}\,u_F^*\wedge u^{F_3}\wedge u^{F_4} - \omega_{F_1F_2F_3}\,u^{F_1}\wedge u^{F_2}\wedge u^{F_3}\,. \tag{6.11}$$

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